Chapter 14

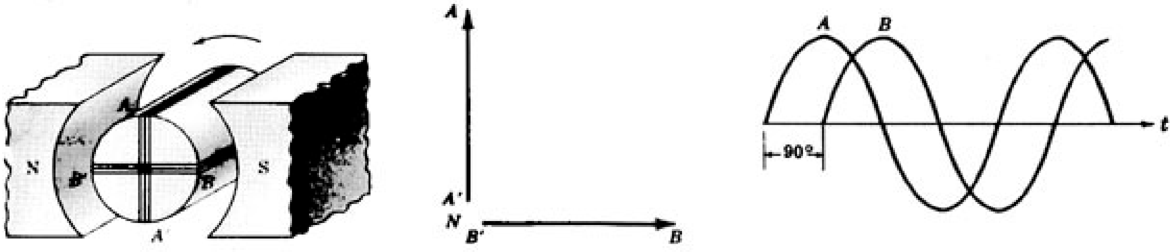
Polyphase Systems

INTRODUCTION

A polyphase system consists of two or more equal voltages with fixed phase differences which supply power to loads connected to the lines. In the two-phase system two equal voltages differ in phase by 90°, while in the three-phase system the voltages have a phase difference of 120°. Systems of six or more phases are sometimes used in polyphase rectifiers to obtain a rectified voltage with low ripple, but three-phase is the system commonly used for generation and transmission of electric power.

TWO-PHASE SYSTEM

Rotation of the pair of perpendicular coils in Fig. 14-1(*a*) in the constant magnetic field results in induced voltages with a fixed 90° phase difference. With equal number of turns of the coils, the phasor and instantaneous voltages have equal magnitudes as shown in their respective diagrams in Figures 14-1(*b*) and (*c*).



(*a*) (*b*) (*c*)

Fig. 14-1. Two-phase System

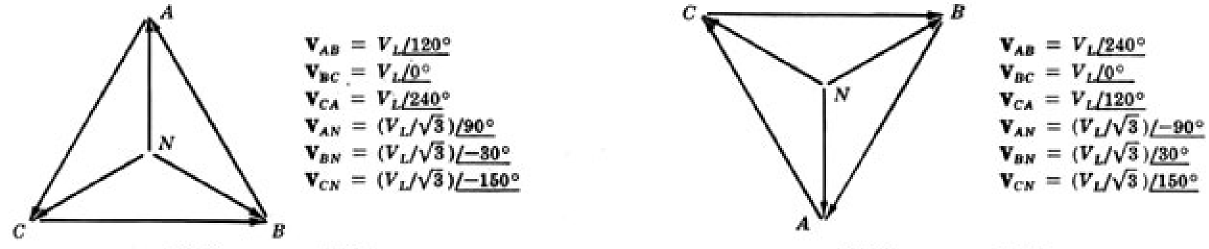
The voltage phasor diagram in Fig. 14-1(*b*) has as a reference $\mathbf{V}_{BN} = V_{\text{coil}}\,\underline{/0^\circ}$ and the voltage $\mathbf{V}_{AN} = V_{\text{coil}}\,\underline{/90^\circ}$. If the coil ends A' and B' are joined as line N, the two-phase system is contained on the three lines A, B and N. The potential between lines A and B exceeds the line to neutral voltages by the factor $\sqrt{2}$ and is obtained from the sum, $\mathbf{V}_{AB} = \mathbf{V}_{AN} + \mathbf{V}_{NB} = V_{\text{coil}}\,\underline{/90^\circ} + V_{\text{coil}}\,\underline{/180^\circ} = \sqrt{2}\,V_{\text{coil}}\,\underline{/135^\circ}$.

THREE-PHASE SYSTEM

The induced voltages in the three equally spaced coils in Fig. 14-2(*a*) below have a phase difference of 120°. The voltage in coil A reaches a maximum first, followed by B and then C for sequence ABC. This sequence is evident from the phasor diagram with its positive rotation counterclockwise where the phasors would pass a fixed point in the order A-B-C-A-B-$C\cdots$, and also from the instantaneous voltage plot of Fig. 14-2(*c*) below where the maxima occur in the same order.

THREE-PHASE SYSTEM VOLTAGES

Selection of one voltage as the reference with a phase angle of zero determines the phase angles of all other voltages in the system. In this chapter $\mathbf{V}_{BC}$ is chosen as reference. The triangles in Figures 14-5(*a*) and (*b*) show all the voltages for sequences *ABC* and *CBA* respectively.



$\mathbf{V}_{AB} = V_L\angle 120°$
$\mathbf{V}_{BC} = V_L\angle 0°$
$\mathbf{V}_{CA} = V_L\angle 240°$
$\mathbf{V}_{AN} = (V_L/\sqrt{3})\angle 90°$
$\mathbf{V}_{BN} = (V_L/\sqrt{3})\angle -30°$
$\mathbf{V}_{CN} = (V_L/\sqrt{3})\angle -150°$

(*a*) Sequence *ABC*

$\mathbf{V}_{AB} = V_L\angle 240°$
$\mathbf{V}_{BC} = V_L\angle 0°$
$\mathbf{V}_{CA} = V_L\angle 120°$
$\mathbf{V}_{AN} = (V_L/\sqrt{3})\angle -90°$
$\mathbf{V}_{BN} = (V_L/\sqrt{3})\angle 30°$
$\mathbf{V}_{CN} = (V_L/\sqrt{3})\angle 150°$

(*b*) Sequence *CBA*

Fig. 14-5

The *system* Line *voltage* is the voltage between any pair of lines, *A* and *B*, *B* and *C* or *C* and *A*. And in the four-wire system the magnitude of the line to neutral voltage is $1/\sqrt{3}$ times line voltage. For example, a three-phase, four-wire, 208 volt, *CBA* system has line voltages of 208 volts and line to neutral voltages of $208/\sqrt{3}$ or 120 volts. Referring to Fig. 14-5(*b*), the phase angles of the voltages are determined. Thus, $\mathbf{V}_{BC} = 208\angle 0°$, $\mathbf{V}_{AB} = 208\angle 240°$, $\mathbf{V}_{CA} = 208\angle 120°$, $\mathbf{V}_{AN} = 120\angle -90°$, $\mathbf{V}_{BN} = 120\angle 30°$ and $\mathbf{V}_{CN} = 120\angle 150°$.

BALANCED THREE-PHASE LOADS

Example 1. A three-phase, three wire, 110 volt, *ABC* system supplies a delta connection of three equal impedances of $5\angle 45°$ ohms. Determine the line currents $\mathbf{I}_A$, $\mathbf{I}_B$ and $\mathbf{I}_C$ and draw the phasor diagram.

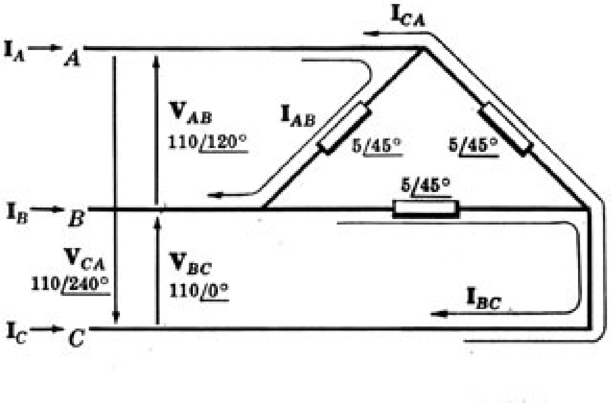


Fig. 14-6

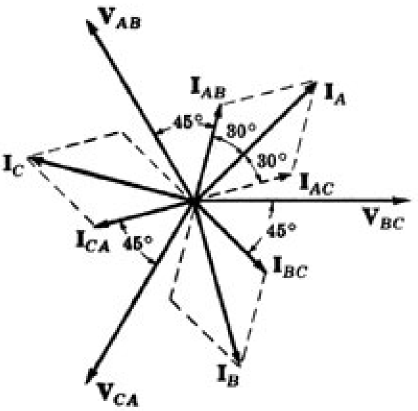


Fig. 14-7

Sketch the circuit and apply the voltages as shown in Fig. 14-6. The positive directions of the line and phase currents are given on the circuit diagram. Then

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}} = \frac{110\angle 120°}{5\angle 45°} = 22\angle 75° = 5.7 + j21.2$$

$$\mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}} = \frac{110\angle 0°}{5\angle 45°} = 22\angle -45° = 15.55 - j15.55$$

$$\mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}} = \frac{110\angle 240°}{5\angle 45°} = 22\angle 195° = -21.2 - j5.7$$

Apply Kirchhoff's current law at each corner of the load and write,

$$\begin{aligned}
\mathbf{I}_A &= \mathbf{I}_{AB} + \mathbf{I}_{AC} = 22\underline{/75^\circ} - 22\underline{/195^\circ} = 38.1\underline{/45^\circ} \\
\mathbf{I}_B &= \mathbf{I}_{BA} + \mathbf{I}_{BC} = -22\underline{/75^\circ} + 22\underline{/-45^\circ} = 38.1\underline{/-75^\circ} \\
\mathbf{I}_C &= \mathbf{I}_{CA} + \mathbf{I}_{CB} = 22\underline{/195^\circ} - 22\underline{/-45^\circ} = 38.1\underline{/165^\circ}
\end{aligned}$$

The phasor diagram in Fig. 14-7 above shows the balanced line currents of 38.1 amperes with 120° phase angles between them.

For a balanced delta-connected load, the line voltage and phase voltage are equal and the line current is $\sqrt{3}$ *times the phase current.*

Example 2. A three-phase, four-wire, 208 volt, *CBA* system serves a balanced wye-connected load with impedances of $20\underline{/-30^\circ}$ ohms. Find the line currents and draw the phasor diagram.

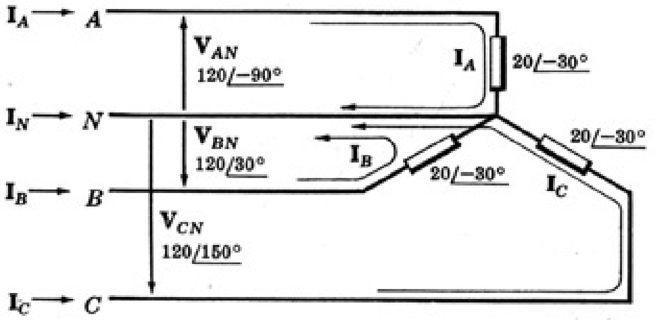


Fig. 14-8

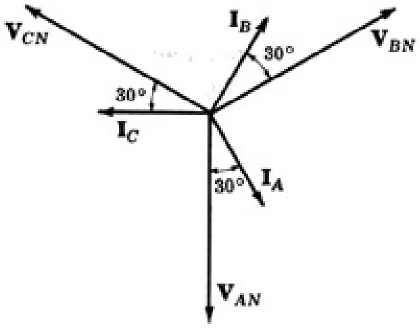


Fig. 14-9

Sketch the circuit and apply the line to neutral voltages using Fig. 14-5(*b*). Select line currents as shown on the diagram in Fig. 14-8 where all currents return through the neutral conductor. Then

$$\begin{aligned}
\mathbf{I}_A &= \frac{\mathbf{V}_{AN}}{\mathbf{Z}} = \frac{120\underline{/-90^\circ}}{20\underline{/-30^\circ}} = 6.0\underline{/-60^\circ} \\
\mathbf{I}_B &= \frac{\mathbf{V}_{BN}}{\mathbf{Z}} = \frac{120\underline{/30^\circ}}{20\underline{/-30^\circ}} = 6.0\underline{/60^\circ} \\
\mathbf{I}_C &= \frac{\mathbf{V}_{CN}}{\mathbf{Z}} = \frac{120\underline{/150^\circ}}{20\underline{/-30^\circ}} = 6.0\underline{/180^\circ}
\end{aligned}$$

Assuming the direction of the neutral current toward the load as positive, we have

$$\mathbf{I}_N = -(\mathbf{I}_A + \mathbf{I}_B + \mathbf{I}_C) = -(6.0\underline{/-60^\circ} + 6.0\underline{/60^\circ} + 6.0\underline{/180^\circ}) = 0$$

The phasor diagram of Fig. 14-9 shows the balanced line currents where each current leads the corresponding line to neutral voltage by the angle on the impedance.

In a balanced wye-connected load the line currents and phase currents are equal, the neutral current is zero, and the line voltage is $\sqrt{3}$ *times the phase voltage, i.e.,* $V_L = \sqrt{3}\,V_P$.

ONE-LINE EQUIVALENT CIRCUIT FOR BALANCED LOADS

According to the Y-Δ transformations of Chapter 12 a set of three equal impedances $\mathbf{Z}_\Delta$ in a delta connection is equivalent to a set of three equal wye-connected impedances $\mathbf{Z}_Y$, where $\mathbf{Z}_Y = (1/3)\mathbf{Z}_\Delta$. Then a more direct computation of the wye circuit is possible for balanced three-phase loads of either type.

The one-line equivalent circuit is one phase of the three-phase, four-wire, wye connected circuit in Fig. 14-10, except that a voltage is used which has the line to neutral magnitude and a phase angle of zero. The line current calculated for this circuit has a phase angle with respect to the phase angle of zero on the voltage. Then the actual line currents $\mathbf{I}_A$, $\mathbf{I}_B$ and $\mathbf{I}_C$ will lead or lag their respective line to neutral voltages by this same phase angle.

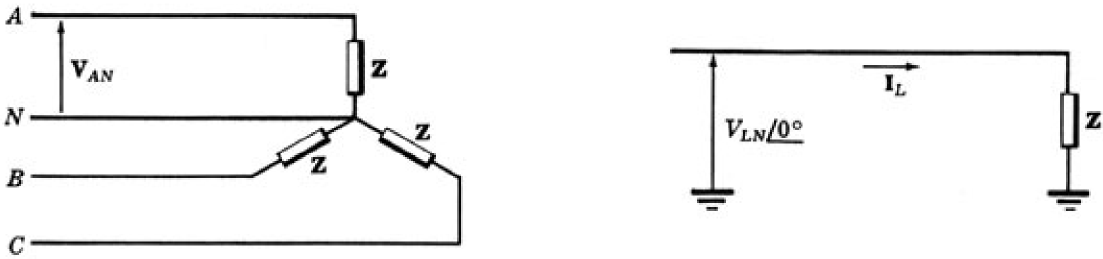


Fig. 14-10. One-line Equivalent Circuit

Example 3. Calculate the line currents of Example 1 by the one-line equivalent method.

Draw the one-line circuit and mark a Δ at the load to show that the actual impedances were in a delta-connection. The impedance of the wye-connected equivalent is

$$\mathbf{Z}_Y = \mathbf{Z}_\Delta/3 = (5/3)\underline{/45^\circ}$$

and the line to neutral voltage is

$$V_{LN} = V_L/\sqrt{3} = 110/\sqrt{3} = 63.5$$

Then the line current is

$$\mathbf{I}_L = \frac{\mathbf{V}_{LN}}{\mathbf{Z}} = \frac{63.5\underline{/0^\circ}}{(5/3)\underline{/45^\circ}} = 38.1\underline{/-45^\circ}$$

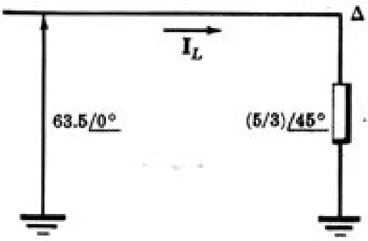


Fig. 14-11

Since this current lags the voltage by 45°, the line currents $\mathbf{I}_A$, $\mathbf{I}_B$ and $\mathbf{I}_C$ lag their respective voltages $\mathbf{V}_{AN}$, $\mathbf{V}_{BN}$ and $\mathbf{V}_{CN}$ by 45°. The angles on these voltages are obtained from the *ABC* triangle of Fig. 14-5(*a*). The line to neutral voltages and the corresponding line currents are tabulated below.

$$\mathbf{V}_{AN} = 63.5\underline{/90^\circ} \qquad \mathbf{I}_A = 38.1\underline{/90^\circ - 45^\circ} = 38.1\underline{/45^\circ}$$

$$\mathbf{V}_{BN} = 63.5\underline{/-30^\circ} \qquad \mathbf{I}_B = 38.1\underline{/-30^\circ - 45^\circ} = 38.1\underline{/-75^\circ}$$

$$\mathbf{V}_{CN} = 63.5\underline{/-150^\circ} \qquad \mathbf{I}_C = 38.1\underline{/-150^\circ - 45^\circ} = 38.1\underline{/-195^\circ}$$

These currents are identical to those obtained in Example 1. If the phase currents in the delta-connected impedances are required, they may be found from $I_P = I_L/\sqrt{3} = 38.1/\sqrt{3} = 22$. The phase angles on these currents are obtained by first setting the phase angles on the line to line voltages and then determining the currents such that they lag by 45°. Hence,

$$\mathbf{V}_{AB} = 110\underline{/120^\circ} \qquad \mathbf{I}_{AB} = 22\underline{/120^\circ - 45^\circ} = 22\underline{/75^\circ}$$

$$\mathbf{V}_{BC} = 110\underline{/0^\circ} \qquad \mathbf{I}_{BC} = 22\underline{/0^\circ - 45^\circ} = 22\underline{/-45^\circ}$$

$$\mathbf{V}_{CA} = 110\underline{/240^\circ} \qquad \mathbf{I}_{CA} = 22\underline{/240^\circ - 45^\circ} = 22\underline{/195^\circ}$$

UNBALANCED DELTA-CONNECTED LOAD

The solution of the unbalanced delta-connected load consists of computing the phase currents and then applying Kirchhoff's current law to the junctions to obtain the three line currents. The line currents will not be equal nor will they have a 120° phase difference as was the case with balanced loads.

Example 4.

A three-phase, three-wire, 240 volt, *ABC* system has a delta-connected load with $\mathbf{Z}_{AB} = 10\underline{/0^\circ}$, $\mathbf{Z}_{BC} = 10\underline{/30^\circ}$ and $\mathbf{Z}_{CA} = 15\underline{/-30^\circ}$. Obtain the three line currents and draw the phasor diagram.

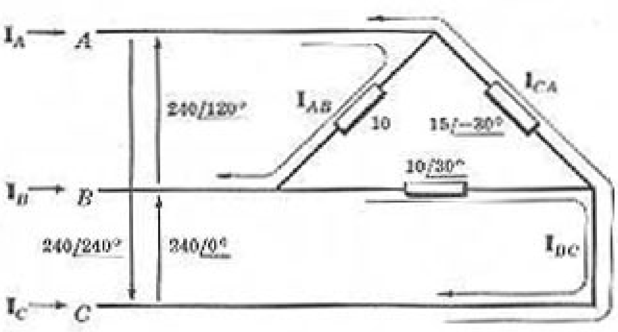


Fig. 14-12

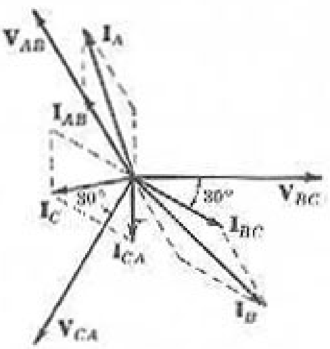


Fig. 14-13

Construct the circuit diagram as in Fig. 14-12, and apply the phasor voltages. Then the phase currents as shown on the diagram are independent and given by

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_{AB}} = \frac{240\underline{/120^\circ}}{10\underline{/0^\circ}} = 24\underline{/120^\circ}, \quad \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_{BC}} = 24\underline{/-30^\circ}, \quad \mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_{CA}} = 16\underline{/270^\circ}$$

Apply Kirchhoff's current law to the junctions of the load and write

$$\mathbf{I}_A = \mathbf{I}_{AB} + \mathbf{I}_{AC} = 24\underline{/120^\circ} - 16\underline{/270^\circ} = 38.7\underline{/108.1^\circ}$$

$$\mathbf{I}_B = \mathbf{I}_{BA} + \mathbf{I}_{BC} = -24\underline{/120^\circ} + 24\underline{/-30^\circ} = 46.4\underline{/-45^\circ}$$

$$\mathbf{I}_C = \mathbf{I}_{CA} + \mathbf{I}_{CB} = 16\underline{/270^\circ} - 24\underline{/-30^\circ} = 21.2\underline{/190.9^\circ}$$

The corresponding phasor diagram is shown in Fig. 14-13.

UNBALANCED FOUR-WIRE, WYE-CONNECTED LOAD

On a four-wire system the neutral conductor will carry a current when the load is unbalanced and the voltage across each of the load impedances remains fixed with the same magnitude as the line to neutral voltage. The line currents are unequal and do not have a 120° phase difference.

Example 5.

A three-phase, four-wire, 208 volt, *CBA* system has a wye-connected load with $\mathbf{Z}_A = 6\underline{/0^\circ}$, $\mathbf{Z}_B = 6\underline{/30^\circ}$ and $\mathbf{Z}_C = 5\underline{/45^\circ}$. Obtain the three line currents and the neutral current. Draw the phasor diagram.

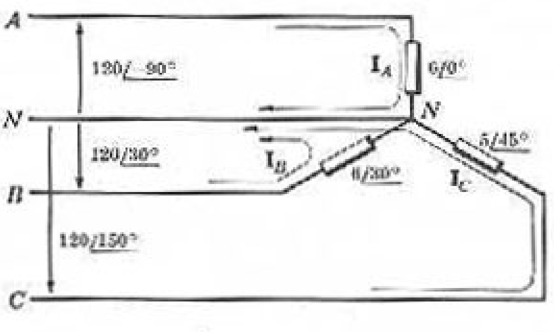


Fig. 14-14

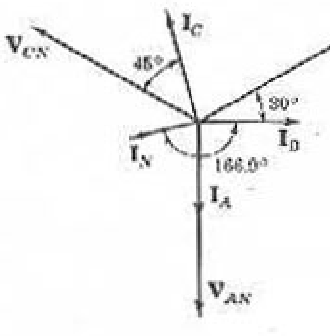


Fig. 14-15

Construct the circuit diagram as in Fig. 14-14 above. Apply the phasor voltages and select the line currents as shown. The currents are independent and given by

$$\mathbf{I}_A = \frac{\mathbf{V}_{AN}}{\mathbf{Z}_A} = \frac{120\underline{/-90^\circ}}{6\underline{/0^\circ}} = 20\underline{/-90^\circ}, \quad \mathbf{I}_B = \frac{\mathbf{V}_{BN}}{\mathbf{Z}_B} = 20\underline{/0^\circ}, \quad \mathbf{I}_C = \frac{\mathbf{V}_{CN}}{\mathbf{Z}_C} = 24\underline{/105^\circ}$$

The neutral conductor contains the sum of the line currents $\mathbf{I}_A$, $\mathbf{I}_B$ and $\mathbf{I}_C$. Then assuming a positive direction of $\mathbf{I}_N$ toward the load,

$$\mathbf{I}_N = -(\mathbf{I}_A + \mathbf{I}_B + \mathbf{I}_C) = -(20\underline{/-90^\circ} + 20\underline{/0^\circ} + 24\underline{/105^\circ}) = 14.1\underline{/-166.9^\circ}$$

The diagram phasor is shown in Fig. 14-15 above.

UNBALANCED THREE-WIRE, WYE-CONNECTED LOAD

With only the three lines A, B and C connected to an unbalanced wye load the common point of the three load impedances is not at the potential of the neutral and is marked "O" instead of N. The voltages across the three impedances can vary considerably from line to neutral magnitude, as shown by the voltage triangle which relates all of the voltages in the circuit. Of particular interest is the displacement of "O" from N, the *displacement neutral voltage.*

Example 6.

A three-phase, three-wire, 208 volt, *CBA* system has a wye-connected load with $\mathbf{Z}_A = 6\underline{/0^\circ}$, $\mathbf{Z}_B = 6\underline{/30^\circ}$ and $\mathbf{Z}_C = 5\underline{/45^\circ}$. Obtain the line currents and the phasor voltage across each impedance. Construct the voltage triangle and determine the displacement neutral voltage, $\mathbf{V}_{ON}$.

Draw the circuit diagram and select mesh currents $\mathbf{I}_1$ and $\mathbf{I}_2$ as shown in Fig. 14-16. Write the corresponding matrix equations of $\mathbf{I}_1$ and $\mathbf{I}_2$ as follows.

$$\begin{bmatrix} 6\underline{/0^\circ} + 6\underline{/30^\circ} & -6\underline{/30^\circ} \\ -6\underline{/30^\circ} & 6\underline{/30^\circ} + 5\underline{/45^\circ} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 208\underline{/240^\circ} \\ 208\underline{/0^\circ} \end{bmatrix}$$

from which $\mathbf{I}_1 = 23.3\underline{/261.1^\circ}$ and $\mathbf{I}_2 = 26.5\underline{/-63.4^\circ}$. Then line currents $\mathbf{I}_A$, $\mathbf{I}_B$ and $\mathbf{I}_C$ directed as shown on the diagram are

$$\mathbf{I}_A = \mathbf{I}_1 = 23.3\underline{/261.1^\circ}$$

$$\begin{aligned} \mathbf{I}_B &= \mathbf{I}_2 - \mathbf{I}_1 = 26.5\underline{/-63.4^\circ} - 23.3\underline{/261.1^\circ} \\ &= 15.45\underline{/-2.5^\circ} \end{aligned}$$

$$\mathbf{I}_C = -\mathbf{I}_2 = 26.5\underline{/116.6^\circ}$$

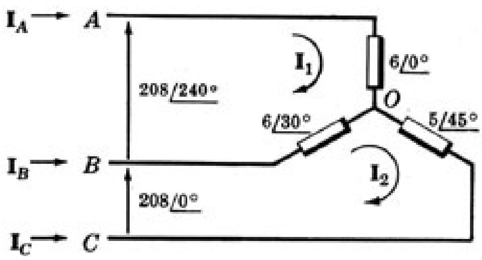


Fig. 14-16

Now the voltages across the three impedances are given by the products of the line currents and the corresponding impedances.

$$\begin{aligned} \mathbf{V}_{AO} &= \mathbf{I}_A\mathbf{Z}_A = 23.3\underline{/261.1^\circ}\,(6\underline{/0^\circ}) = 139.8\underline{/261.1^\circ} \\ \mathbf{V}_{BO} &= \mathbf{I}_B\mathbf{Z}_B = 15.45\underline{/-2.5^\circ}\,(6\underline{/30^\circ}) = 92.7\underline{/27.5^\circ} \\ \mathbf{V}_{CO} &= \mathbf{I}_C\mathbf{Z}_C = 26.5\underline{/116.6^\circ}\,(5\underline{/45^\circ}) = 132.5\underline{/161.6^\circ} \end{aligned}$$

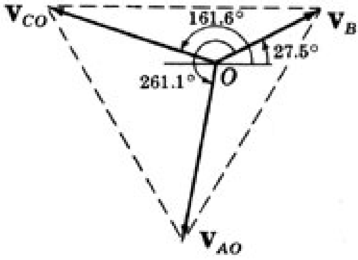


Fig. 14-17

The phasor diagram of these three voltages shown in Fig. 14-17 forms an equilateral triangle. In Fig. 14-18 this triangle is redrawn and the neutral is added, thus showing the displacement neutral voltage $\mathbf{V}_{ON}$. This voltage may be computed using any of the three points A, B or C and following the conventional double subscript notation. Using point A, we obtain

$$\begin{aligned} \mathbf{V}_{ON} &= \mathbf{V}_{OA} + \mathbf{V}_{AN} = -139.8\underline{/261.1^\circ} + 120\underline{/-90^\circ} \\ &= 28.1\underline{/39.8^\circ} \end{aligned}$$

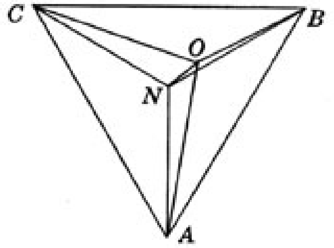


Fig. 14-18

DISPLACEMENT NEUTRAL METHOD, UNBALANCED THREE-WIRE WYE LOAD

In Example 6 the displacement neutral voltage $\mathbf{V}_{ON}$ was obtained in terms of load voltages. If we determine a relation for $\mathbf{V}_{ON}$ independent of the load voltages, then the required currents and voltages of Example 6 are obtained more directly as shown in Example 7.

To obtain the displacement neutral voltage write the line currents in terms of the load voltages and load admittances.

$$\mathbf{I}_A = \mathbf{V}_{AO}\mathbf{Y}_A, \quad \mathbf{I}_B = \mathbf{V}_{BO}\mathbf{Y}_B, \quad \mathbf{I}_C = \mathbf{V}_{CO}\mathbf{Y}_C \qquad (1)$$

Now apply Kirchhoff's current law at point O in Fig. 14-19 and write

$$\mathbf{I}_A + \mathbf{I}_B + \mathbf{I}_C \;=\; 0 \qquad (2)$$

or

$$\mathbf{V}_{AO}\mathbf{Y}_A + \mathbf{V}_{BO}\mathbf{Y}_B + \mathbf{V}_{CO}\mathbf{Y}_C \;=\; 0 \qquad (3)$$

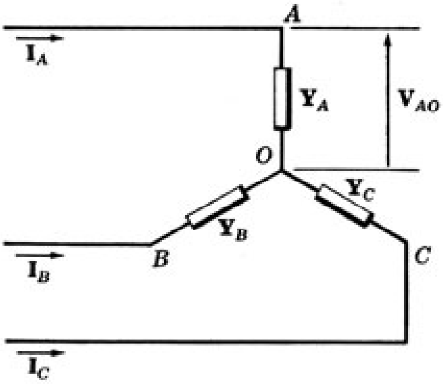


Fig. 14-19

Referring to the diagram of Fig. 14-18, express the voltages $\mathbf{V}_{AO}$, $\mathbf{V}_{BO}$ and $\mathbf{V}_{CO}$ in terms of their two component voltages, i.e.,

$$\mathbf{V}_{AO} \;=\; \mathbf{V}_{AN} + \mathbf{V}_{NO} \qquad \mathbf{V}_{BO} \;=\; \mathbf{V}_{BN} + \mathbf{V}_{NO} \qquad \mathbf{V}_{CO} \;=\; \mathbf{V}_{CN} + \mathbf{V}_{NO} \qquad (4)$$

Substituting the expressions of (*4*) into (*3*) we obtain

$$(\mathbf{V}_{AN} + \mathbf{V}_{NO})\mathbf{Y}_A \;+\; (\mathbf{V}_{BN} + \mathbf{V}_{NO})\mathbf{Y}_B \;+\; (\mathbf{V}_{CN} + \mathbf{V}_{NO})\mathbf{Y}_C \;=\; 0 \qquad (5)$$

from which

$$\mathbf{V}_{ON} \;=\; \frac{\mathbf{V}_{AN}\mathbf{Y}_A + \mathbf{V}_{BN}\mathbf{Y}_B + \mathbf{V}_{CN}\mathbf{Y}_C}{\mathbf{Y}_A + \mathbf{Y}_B + \mathbf{Y}_C} \qquad (6)$$

The voltages $\mathbf{V}_{AN}$, $\mathbf{V}_{BN}$ and $\mathbf{V}_{CN}$ in equation (*6*) are obtained from the triangle of Fig. 14-5 for the sequence given in the problem. And admittances $\mathbf{Y}_A$, $\mathbf{Y}_B$ and $\mathbf{Y}_C$ are the reciprocals of the load impedances $\mathbf{Z}_A$, $\mathbf{Z}_B$ and $\mathbf{Z}_C$. Therefore, since all of the terms in (*6*) are either given or readily obtained, the displacement neutral voltage may be computed and then used to determine the line currents.

Example 7.

Obtain the line currents and load voltages in Example 6 by the displacement neutral method.

Referring to Fig. 14-20, the equation for the displacement neutral voltage is

$$\mathbf{V}_{ON} \;=\; \frac{\mathbf{V}_{AN}\mathbf{Y}_A + \mathbf{V}_{BN}\mathbf{Y}_B + \mathbf{V}_{CN}\mathbf{Y}_C}{\mathbf{Y}_A + \mathbf{Y}_B + \mathbf{Y}_C}$$

where

$$\begin{aligned}
\mathbf{Y}_A &= 1/(6\underline{/0^\circ}) &&= .1667\underline{/0^\circ} &&= .1667\\
\mathbf{Y}_B &= 1/(6\underline{/30^\circ}) &&= .1667\underline{/-30^\circ} &&= .1443 - j\,.0833\\
\mathbf{Y}_C &= 1/(5\underline{/45^\circ}) &&= .20\underline{/-45^\circ} &&= \underline{.1414 - j\,.1414}\\
& && \mathbf{Y}_A + \mathbf{Y}_B + \mathbf{Y}_C &&= .4524 - j\,.2247\\
& && &&= .504\underline{/-26.5^\circ}
\end{aligned}$$

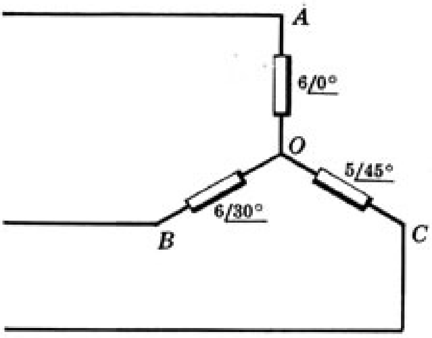


Fig. 14-20

and

$$\begin{aligned}
\mathbf{V}_{AN}\mathbf{Y}_A &= 120\underline{/-90^\circ}\,(.1667\underline{/0^\circ}) &&= 20\underline{/-90^\circ} &&= \quad -j20\\
\mathbf{V}_{BN}\mathbf{Y}_B &= 120\underline{/30^\circ}\,(.1667\underline{/-30^\circ}) &&= 20\underline{/0^\circ} &&= 20\\
\mathbf{V}_{CN}\mathbf{Y}_C &= 120\underline{/150^\circ}\,(.20\underline{/-45^\circ}) &&= 24\underline{/105^\circ} &&= \underline{-6.2 + j23.2}\\
& \mathbf{V}_{AN}\mathbf{Y}_A + \mathbf{V}_{BN}\mathbf{Y}_B + \mathbf{V}_{CN}\mathbf{Y}_C && &&= 13.8 + \;j3.2 \;=\; 14.1\underline{/13.1^\circ}
\end{aligned}$$

Then

$$\mathbf{V}_{ON} \;=\; 14.1\underline{/13.1^\circ}/.504\underline{/-26.5^\circ} \;=\; 28.0\underline{/39.6^\circ}$$

Voltages $\mathbf{V}_{AO}$, $\mathbf{V}_{BO}$ and $\mathbf{V}_{CO}$ are obtained using $\mathbf{V}_{NO}$ and the appropriate line to neutral voltage.

$$\mathbf{V}_{AO} = \mathbf{V}_{AN} + \mathbf{V}_{NO} = 120\underline{/-90^\circ} - 28.0\underline{/39.6^\circ} = 139.5\underline{/261.1^\circ}$$

$$\mathbf{V}_{BO} = \mathbf{V}_{BN} + \mathbf{V}_{NO} = 120\underline{/30^\circ} - 28.0\underline{/39.6^\circ} = 92.5\underline{/27.1^\circ}$$

$$\mathbf{V}_{CO} = \mathbf{V}_{CN} + \mathbf{V}_{NO} = 120\underline{/150^\circ} - 28.0\underline{/39.6^\circ} = 132.5\underline{/161.45^\circ}$$

The line currents are readily obtained from the voltages and the corresponding load admittances:

$$\mathbf{I}_A = \mathbf{V}_{AO}\mathbf{Y}_A = 139.5\underline{/261.1^\circ}\,(.1667\underline{/0^\circ}) = 23.2\underline{/261.1^\circ}$$

$$\mathbf{I}_B = \mathbf{V}_{BO}\mathbf{Y}_B = 92.5\underline{/27.1^\circ}\,(.1667\underline{/-30^\circ}) = 15.4\underline{/-2.9^\circ}$$

$$\mathbf{I}_C = \mathbf{V}_{CO}\mathbf{Y}_C = 132.5\underline{/161.45^\circ}\,(.20\underline{/-45^\circ}) = 26.5\underline{/116.45^\circ}$$

The above currents and voltages compare favorably with the results of Example 6.

POWER IN BALANCED THREE-PHASE LOADS

Since the phase impedances of balanced wye or delta loads contain equal currents, the phase power is one-third of the total power. The voltage across impedance $\mathbf{Z}_\Delta$ in Fig. 14-21(*a*) is *line voltage* and the current is *phase current*. The angle between the voltage and current is the angle on the impedance. Then the phase power is

$$P_P = V_L I_P \cos\theta \tag{7}$$

and the total power is

$$P_T = 3\,V_L I_P \cos\theta \tag{8}$$

Since $I_L = \sqrt{3}\,I_P$ in balanced delta-connected loads,

$$P_T = \sqrt{3}\,V_L I_L \cos\theta \tag{9}$$

The wye-connected impedances of Fig. 14-21(*b*) contain the *line currents*, and the voltage across $\mathbf{Z}_Y$ is a *phase voltage*. The angle between them is the angle on the impedance. Then the phase power is

$$P_P = V_P I_L \cos\theta \tag{10}$$

and the total power is

$$P_T = 3\,V_P I_L \cos\theta \tag{11}$$

Since $V_L = \sqrt{3}\,V_P$,

$$P_T = \sqrt{3}\,V_L I_L \cos\theta \tag{12}$$

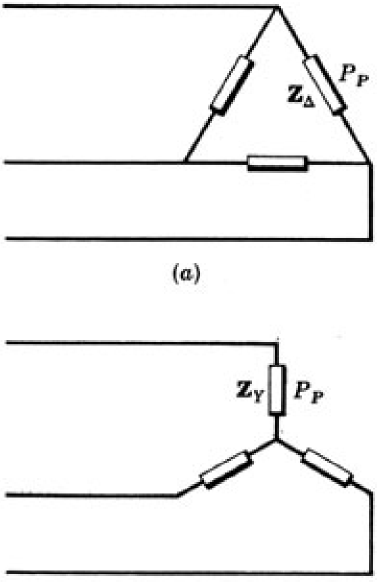


Fig. 14-21

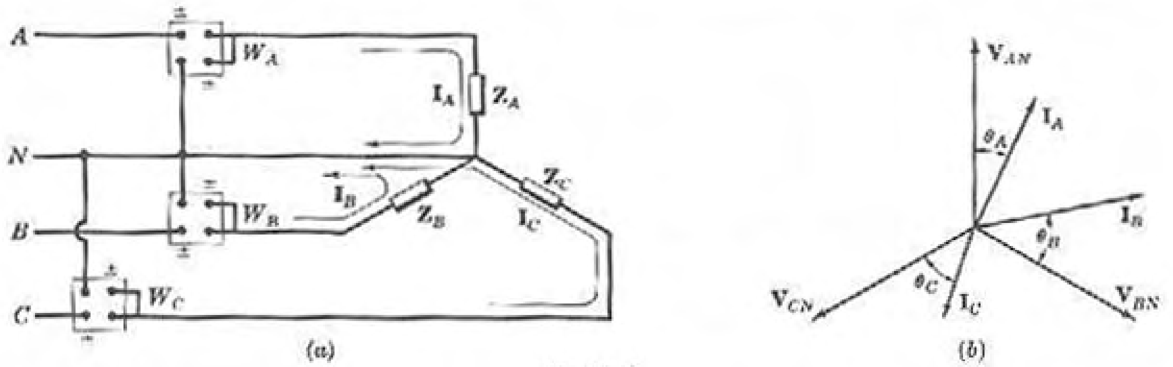
Since equations (*9*) and (*12*) are identical, the total power in any balanced three-phase load is given by $\sqrt{3}\,V_L I_L \cos\theta$ where θ is the angle on the load impedance or the angle on an equivalent impedance in the case where several balanced loads are served from the same system.

The total volt-amperes S_T and the total reactive power Q_T are related to P_T in Chapter 7. Therefore a balanced three-phase load has the power, apparent power, and reactive power given by

$$P_T = \sqrt{3}\,V_L I_L \cos\theta \qquad S_T = \sqrt{3}\,V_L I_L \qquad Q_T = \sqrt{3}\,V_L I_L \sin\theta \tag{13}$$

WATTMETERS AND FOUR-WIRE WYE LOADS

A wattmeter is an instrument with a potential coil and a current coil so arranged that its deflection is proportional to $VI \cos\theta$ where θ is the angle between the voltage and current. A four-wire, wye-connected load requires three wattmeters with one meter in each line as shown in Fig. 14-22(*a*).



(*a*) (*b*)

Fig. 14-22

The phasor diagram in Fig. 14-22(*b*) assumes a lagging current in phase *A* and leading currents in phases *B* and *C* with phase angles θ_A, θ_B and θ_C respectively. Then the wattmeter readings are

$$W_A = V_{AN} I_A \cos \measuredangle^{AN}_{A}, \qquad W_B = V_{BN} I_B \cos \measuredangle^{BN}_{B}, \qquad W_C = V_{CN} I_C \cos \measuredangle^{CN}_{C} \tag{14}$$

where $\measuredangle^{AN}_{A}$ indicates the angle between $\mathbf{V}_{AN}$ and $\mathbf{I}_A$. Wattmeter W_A reads the power in phase *A* and wattmeters W_B and W_C the power in phases *B* and *C* respectively. The total power is

$$P_T = W_A + W_B + W_C \tag{15}$$

TWO-WATTMETER METHOD

The total power in a three-phase, three-wire load is given by the sum of the readings on two wattmeters connected in any two lines with their potential coils connected to the third line as shown in Fig. 14-23. The readings of the meters are

$$W_A = V_{AB} I_A \cos \measuredangle^{AB}_{A} \qquad \text{and} \qquad W_C = V_{CB} I_C \cos \measuredangle^{CB}_{C} \tag{16}$$

Applying Kirchhoff's current law to junctions *A* and *C* of the delta load, we obtain

$$\mathbf{I}_A = \mathbf{I}_{AB} + \mathbf{I}_{AC} \qquad \text{and} \qquad \mathbf{I}_C = \mathbf{I}_{CA} + \mathbf{I}_{CB} \tag{17}$$

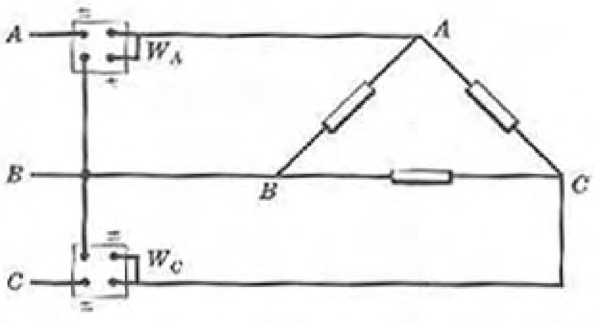


Fig. 14-23

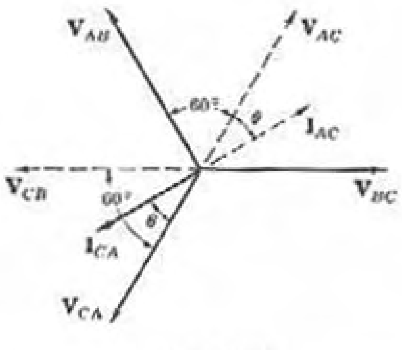


Fig. 14-24

Substituting the expressions for $\mathbf{I}_A$ and $\mathbf{I}_C$ given in (*17*) into the wattmeter equations (*16*), we obtain

$$\begin{aligned} W_A &= V_{AB}I_{AB}\cos \angle^{AB}_{AB} + V_{AB}I_{AC}\cos\angle^{AB}_{AC} \\ W_C &= V_{CB}I_{CA}\cos \angle^{CB}_{CA} + V_{CB}I_{CB}\cos\angle^{CB}_{CB} \end{aligned} \tag{18}$$

The terms $V_{AB}I_{AB}\cos\angle^{AB}_{AB}$ and $V_{CB}I_{CB}\cos\angle^{CB}_{CB}$ are immediately recognized as the power in phases *AB* and *CB* of the load. The two remaining terms contain $V_{AB}I_{AC}$ and $V_{CB}I_{CA}$ which can now be written $V_L I_{AC}$ since both V_{AB} and V_{CB} are line voltages and $I_{AC} = I_{CA}$. To identify these two terms, construct the phasor diagram of Fig. 14-24 above where current $\mathbf{I}_{AC}$ is assumed to lag $\mathbf{V}_{AC}$ by θ.

From the phasor diagram,

$$\angle^{AB}_{AC} = 60° + \theta \qquad \text{and} \qquad \angle^{CB}_{CA} = 60° - \theta \tag{19}$$

Now add the two remaining wattmeter terms from (*18*) and substitute $(60° + \theta)$ and $(60° - \theta)$ for $\angle^{AB}_{AC}$ and $\angle^{CB}_{CA}$ respectively.

$$V_L I_{AC}\cos(60° + \theta) + V_L I_{AC}\cos(60° - \theta) \tag{20}$$

Since $\cos(x \pm y) = \cos x\cos y \mp \sin x \sin y$, we write

$$V_L I_{AC}(\cos 60°\cos\theta - \sin 60°\sin\theta + \cos 60°\cos\theta + \sin 60°\sin\theta) \tag{21}$$

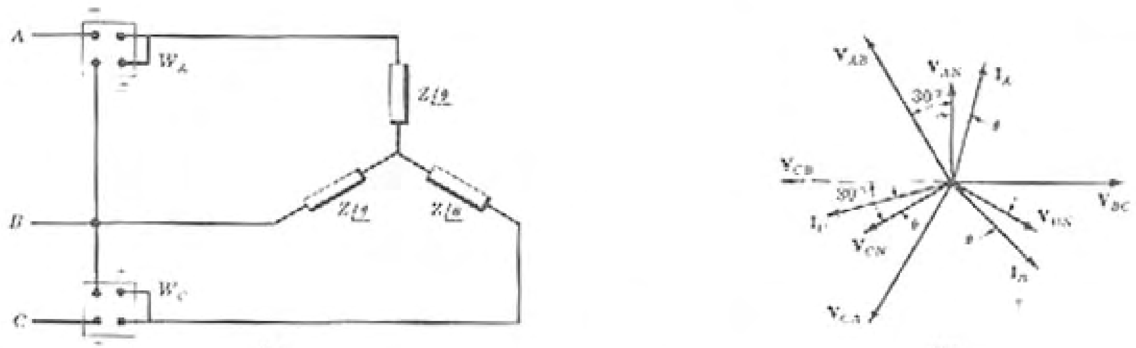
or

$$V_L I_{AC}\cos\theta \tag{22}$$

which is the power in the remaining phase of the load, phase *AC*. Thus we find that two wattmeters indicate the total power in a delta-connected load. The two-wattmeter method for a wye-connected load is left as an exercise for the reader.

TWO-WATTMETER METHOD APPLIED TO BALANCED LOADS

To show the application of the two-wattmeter method to balanced loads, consider the wye-connection of the three equal impedances shown in Fig. 14-25(*a*). The phasor diagram is drawn in Fig. 14-25(*b*) for the *ABC* sequence with the assumption of a lagging current with phase angle θ.



(*a*) (*b*)

Fig. 14-25

Now with the wattmeters in lines *A* and *C*, their readings are

$$W_A = V_{AB}I_A\cos\angle^{AB}_{A} \qquad \text{and} \qquad W_C = V_{CB}I_C\cos\angle^{CB}_{C} \tag{23}$$

From the phasor diagram,

$$\angle^{AB}_{A} = 30° + \theta \qquad \text{and} \qquad \angle^{CB}_{C} = 30° - \theta \tag{24}$$

Substituting *(24)* in *(23)*, we have

$$W_A = V_{AB}I_A\cos(30° + \theta) \quad \text{and} \quad W_C = V_{CB}I_C\cos(30° - \theta) \tag{25}$$

When the two-wattmeter method is used on a balanced load, the wattmeter readings are $V_L I_L \cos(30° + \theta)$ and $V_L I_L \cos(30° - \theta)$ where θ is the angle on the impedance. The two readings can be used to determine the angle θ.

Writing the expression for W_1 and using the cosine of the sum of two angles, we obtain

$$W_1 = V_L I_L(\cos 30° \cos\theta - \sin 30° \sin\theta) \tag{26}$$

Similarly,

$$W_2 = V_L I_L(\cos 30° \cos\theta + \sin 30° \sin\theta) \tag{27}$$

Then the sum $W_1 + W_2 = \sqrt{3}\,V_L I_L\cos\theta$ and the difference $W_2 - W_1 = V_L I_L \sin\theta$, from which

$$\tan\theta = \sqrt{3}\left(\frac{W_2 - W_1}{W_1 + W_2}\right) \tag{28}$$

Thus the tangent of the angle on **Z** is $\sqrt{3}$ times the ratio of the difference between the two wattmeter readings and their sum. With no knowledge of the lines in which the meters are located nor of the system sequence, it is not possible to distinguish between $+\theta$ and $-\theta$. However, when both the sequence and meter location are known, the sign can be fixed by the following expressions. For sequence *ABC*,

$$\tan\theta = \sqrt{3}\,\frac{W_A - W_B}{W_A + W_B} = \sqrt{3}\,\frac{W_B - W_C}{W_B + W_C} = \sqrt{3}\,\frac{W_C - W_A}{W_C + W_A} \tag{29}$$

and for *CBA*,

$$\tan\theta = \sqrt{3}\,\frac{W_B - W_A}{W_B + W_A} = \sqrt{3}\,\frac{W_C - W_B}{W_C + W_B} = \sqrt{3}\,\frac{W_A - W_C}{W_A + W_C} \tag{30}$$

Solved Problems

14.1. Show that the line voltage V_L in the three-phase system is $\sqrt{3}$ times the line to neutral voltage V_p.

In Fig. 14-26, the voltages of the three-phase system are shown on an equilateral triangle in which the length of a side is proportional to the line voltage V_L and where the neutral point N is at the center of the triangle.

The line to neutral voltage has a horizontal projection, $V_P \cos 30°$ or $V_P\sqrt{3}/2$. Since the base is the sum of two such projections, it follows that

$$V_L = 2(V_P\sqrt{3}/2) = \sqrt{3}\,V_P$$

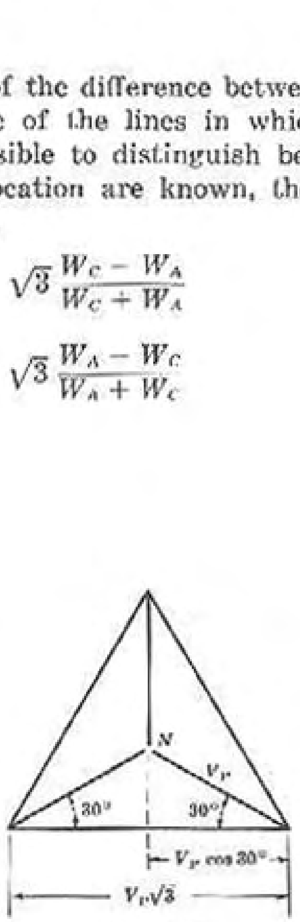


Fig. 14-26

14.2. Compute the full-load coil currents for both delta and wye connected, three-phase alternators with a rating of 25 kva at 480 volts.

With the wye connection the line current and coil current have the same magnitude. For a balanced three-phase system,

$$\text{kva} = \sqrt{3}\,V_L I_L \times 10^{-3} \quad \text{and} \quad I_L = \frac{\text{kva}}{\sqrt{3}\,V_L \times 10^{-3}} = \frac{25}{\sqrt{3}\,(480 \times 10^{-3})} = 30.1$$

The delta-connected alternator with the same kva rating also has full-load line currents of 30.1 amp. The coil currents are $I_L/\sqrt{3}$. Then $I_{\text{coil}} = 30.1/\sqrt{3} = 17.35$ amp.